An introduction to p-adic and characteristic-p cohomology theories: miracles and pathologies

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1 Introduction

Foreword: p-adic cohomology theories are always surrounded by lots of technicalities. I realized while preparing this talk that I won't be able to tell you everything about all (or indeed any) of the classical cohomology theories. Instead I will try to give you a general taste of what they are like, and then go into more detail on de Rham cohomology and its relation to Hodge cohomology. Finally I will give some examples of the extent to which things can misbehave in characteristic-p and p-adic situations.

2 Rundown of classical cohomology theories

The story of p-adic cohomologies in algebraic geometry begins with André Weil, who in 1949 proposed his conjectures on point counts of algebraic varieties over finite fields. His proposed method of attacking the conjectures, which was ultimately carried out by Dwork, Grothendieck, Artin, Deligne, and others, was to obtain them as formal consequences of the existence of a certain type of cohomology theory, now known as a Weil cohomology theory. Such a cohomology theory should behave like singular cohomology for $\mathbb{C}$-varieties, but should take varieties over finite fields as input. This was eventually accomplished by étale and later crystalline cohomology, which are both defined over p-adic rings.

Fix an arbitrary base field k and a coefficient field K of characteristic 0. A K-valued Weil cohomology theory for k-varieties consists of a contravariant functor H^* from the category of smooth projective varieties over k to the category of finite-dimensional graded-commutative K-algebras, along with some extra data, satisfying some formal properties. For us the most important of these extra properties is Poincaré duality, which says that $H^i(X) = 0$ for $i \notin [0, 2n]$ (for $n = \dim X$), $H^0 \cong H^{2n} \cong K$, and the multiplication maps $H^i(X) \times H^{2n-i}(X) \to H^{2n}(X) \cong K$ are perfect pairings.[1]

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[1] A Weil cohomology theory is also expected to satisfy the Künneth formula $H^*(X) \otimes H^*(Y) \cong H^*(X \times_k Y)$, and to receive a cycle class map from the Chow ring—i.e. a closed subvariety $Z \subset X$ of codimension d must

Besides singular cohomology for (the analytification of) varieties over $\mathbb{C}$, there are basically three classical Weil cohomology theories:

name	étale	de Rham	crystalline
base ring	k a field	any	k a perfect field of char p
coefficient ring	$\mathbb{Z}_\ell$	= base ring	$W(k)$
Weil coh if:	$\ell \neq \operatorname{char} k$, $\otimes \mathbb{Q}$	over k, $\operatorname{char} k = 0$	$\otimes \mathbb{Q}$
idea of construction	étale site	hypercoh of $\Omega^\bullet_{X/k}$	crystalline site, or $W\Omega^\bullet_{X/k}$
extra structure	action of G_k	Hodge filtration	semilinear Frobenius
intuition	$\approx$ singular	like analytic dR	H^*_{dR}(smooth lift to $W(k)$), if $\exists$

Remark: We can get all three of these in the same picture if we start with a variety over a p-adic integer ring $\mathcal{O}_L$, such as $\mathbb{Z}_p$. Namely, we can take the p-adic étale cohomology of the generic fiber (base = L, coefficients = $\mathbb{Z}_p$), the de Rham cohomology of the full integral model (base = coefficients = $\mathcal{O}_L$), and the crystalline cohomology of the special fiber (base = $k = \mathcal{O}_L/\mathfrak{m}$, coefficients = $W(k)$). Time permitting, I will say something later about how these compare.

I won't have time to say much about all of these. Étale and crystalline cohomology are both things that one can spend years learning about. Instead, I'll just talk about the most accessible of the three, de Rham cohomology, and explain its relationship to Hodge cohomology. Then I'll mention some pathologies that can happen in characteristic-p and p-adic situations.

2.1 Non-example: sheaf cohomology of $\mathcal{O}_X$

One interesting cohomology theory which is *not* a Weil cohomology is the sheaf cohomology $H^i(X, \mathcal{O}_X)$. Recall that sheaf cohomology is defined as the derived functors of the global sections functor $H^0(X, -) = \Gamma(X, -)$, and it can be computed in principle by an injective resolution of the sheaf in question. (In practice, we usually calculate it using long exact sequences, or if absolutely necessary by a Čech cover.) Namely, if $0 \to \mathcal{F} \to I_0 \to I_1 \to \cdots$ is an injective resolution[2], then $H^*(X, \mathcal{F})$ is the cohomology of the complex $0 \to \Gamma(X, I_0) \to \Gamma(X, I_1) \to \cdots$. Remember this for later:

$$\begin{aligned} \text{sheaf cohomology} &:= \text{derived functors of } \Gamma && (1) \\ &:= \text{cohomology of } (\Gamma \text{ applied to an injective resolution of } \mathcal{F}). && (2) \end{aligned}$$

What's "wrong" with sheaf cohomology of $\mathcal{O}_X$? First of all, if X is n-dimensional, then sheaf cohomology vanishes above cohomological degree n, instead of $2n$. Moreover, the top cohomology group need not be 1-dimensional; it is often bigger and often trivial. So there is no hope for Poincaré duality.

induce a class in $H^{2d}(X)$, with various compatibilities. Some authors also require Weil cohomologies to satisfy the Lefschetz theorems.

[2] An object in an abelian category is *injective* if all maps to it extend along injections. Think of $\mathbb{Q}$ as a $\mathbb{Z}$-module. In the category of abelian sheaves, they are constructed as products of skyscraper sheaves—not something you'll ever want to work with.

The duality that *is* enjoyed by sheaf cohomology is Serre duality, which (for X/k smooth projective of dimension n and $\mathcal{F}$ a vector bundle on X) takes the form $H^i(X, \mathcal{F})^\vee \cong H^{n-i}(X, \mathcal{F}^\vee \otimes \omega_X)$. But this relates the cohomology of $\mathcal{O}_X$ to that of ω_X, not to itself. Still, this is related to Poincaré duality (for de Rham cohomology), and we will see it again soon.

2.2 de Rham

For X/k smooth of dimension n (k an arbitrary field), recall that the sheaf of differentials $\Omega^1_{X/k}$ is a vector bundle of rank n. The bundle $\Omega^i_{X/k}$ of i-forms is defined to be its ith exterior power, and these come equipped with k-linear (not $\mathcal{O}_X$-linear) maps $d : \Omega^i \to \Omega^{i+1}$ forming a complex

$$\mathcal{O}_X = \Omega^0_{X/k} \to \Omega^1_{X/k} \to \Omega^2_{X/k} \to \cdots \to \Omega^n \to 0, \tag{3}$$

known as the de Rham complex $\Omega^\bullet_{X/k}$. The de Rham cohomology of X is defined as the hypercohomology of this complex.

What does hypercohomology mean? The point is that the literal cohomology sheaves of $\Omega^\bullet_{X/k}$, or the cohomology groups of $\Gamma(X, \Omega^\bullet_{X/k})$, aren't the right objects to consider, because each Ω^i may itself have nontrivial sheaf cohomology. So "hypercohomology" means "replace it with an injective resolution, apply the global sections functor Γ, and then take cohomology"—just like the sheaf cohomology from before, but this time we're starting with something that is already a complex. In fact, this is exactly the meaning of the right derived functors $R^i\Gamma(X, -)$.

In principle, one could compute hypercohomology as follows. First choose an injective resolution of each Ω^i. With some care, one can do this compatibly for all i: that is, we can construct a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_{02} & \longrightarrow & I_{12} & \longrightarrow & I_{22} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_{01} & \longrightarrow & I_{11} & \longrightarrow & I_{21} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & I_{00} & \longrightarrow & I_{10} & \longrightarrow & I_{20} & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^2 & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the I_{ij} are injective sheaves of k-vector spaces, the rows are complexes, and the columns are exact. Reading down the diagonals of this diagram (and putting alternating signs along the rows or columns to make the diagram anti-commutative instead of commutative), we get a quasi-isomorphism from $\Omega^\bullet_{X/k}$ to a complex of injective sheaves:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega^0 & \longrightarrow & \Omega^1 & \longrightarrow & \Omega^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & I_{00} & \xrightarrow{\pm} & I_{01} \oplus I_{10} & \xrightarrow{\pm} & I_{02} \oplus I_{11} \oplus I_{20} & \xrightarrow{\pm} & \cdots
\end{array}$$

This bottom complex is a good substitute for $\Omega^\bullet_{X/k}$ in the sense that it has the same cohomology sheaves, but the sheaf cohomology of its terms is trivial. The hypercohomology $\mathbb{H}^*(\Omega^\bullet_{X/k})$ is defined as the cohomology of Γ(this complex).

Of course, I would never recommend actually computing an injective resolution of anything. There's a better way, given by the machinery of spectral sequences. Going back to the big double complex from a little while ago, let's delete the $\Omega^\bullet$ row and apply Γ to everything:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \iddots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I_{02}) & \longrightarrow & \Gamma(I_{12}) & \longrightarrow & \Gamma(I_{22}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I_{01}) & \longrightarrow & \Gamma(I_{11}) & \longrightarrow & \Gamma(I_{21}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \Gamma(I_{00}) & \longrightarrow & \Gamma(I_{10}) & \longrightarrow & \Gamma(I_{20}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

(This is called the E_0 page of the spectral sequence.) We want the cohomology of the total complex of this; i.e. the complex whose terms are the successive antidiagonals of the diagram. Let's first calculate the cohomology just in the vertical direction:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^2(\mathcal{O}_X) & \longrightarrow & H^2(\Omega^1_{X/k}) & \longrightarrow & H^2(\Omega^2_{X/k}) & \longrightarrow & \cdots \\
 & & & & & & & & \\
0 & \longrightarrow & H^1(\mathcal{O}_X) & \longrightarrow & H^1(\Omega^1_{X/k}) & \longrightarrow & H^1(\Omega^2_{X/k}) & \longrightarrow & \cdots \\
 & & & & & & & & \\
0 & \longrightarrow & H^0(\mathcal{O}_X) & \longrightarrow & H^0(\Omega^1_{X/k}) & \longrightarrow & H^0(\Omega^2_{X/k}) & \longrightarrow & \cdots
\end{array}$$

(This is the E_1 page.) These are familiar objects—the sheaf cohomology groups of the sheaves of differentials—and they can actually be computed in practice. This is great, because if we compute these, we can avoid thinking about any actual injective resolutions, and just start the calculation here.

By commutativity of the diagram, we get induced maps in the horizontal direction, forming a complex in each row. Then we take cohomology of the rows, and by some diagram chase we end up getting maps in the $(2,-1)$ direction, then $(3,-2)$, and so on. This continues forever. But each position in the diagram eventually stabilizes, because the maps to and from it eventually have their other endpoints outside of the first quadrant. The machinery of spectral sequences tells us that the objects on the resulting E_∞ page are the successive quotients of a filtration of the hypercohomology $H^i_{dR}(X/k)$. This is called the *Hodge filtration* on de Rham cohomology, and the spectral sequence producing it is called the *Hodge-de Rham spectral sequence*.

Let's make a few more observations, assuming that X/k is smooth projective of dimension n. Recall our statement of Serre duality: for $\mathcal{F}$ a vector bundle on X, we have $H^i(X,\mathcal{F})^\vee \cong H^{n-i}(X,\mathcal{F}^\vee \otimes \omega_X)$. For $\mathcal{F} = \Omega^i_{X/k}$, we have $\mathcal{F}^\vee \otimes \omega_X = \Omega^{n-i}_{X/k}$, since the sheaves of differentials in complementary dimensions admit a perfect pairing up to $\omega_X = \Omega^n_{X/k}$. So Serre duality says that the E_1-page of the spectral sequence is symmetric with respect to a 180-degree rotation.[3] This symmetry gives rise to Poincaré duality for de Rham cohomology. In particular, we have $H^0_{dR}(X/k) = H^0(\mathcal{O}_X) = k$, $H^{2n}_{dR}(X/k) = H^n(\omega_X) = H^0(\mathcal{O}_X)^\vee = k$, and $H^{>2n}_{dR}(X/k) = 0$.

At this point it may be useful to write down some Hodge diamonds; i.e. arrays of integers recording the dimensions of the Hodge cohomology groups $H^q(X, \Omega^p_{X/k})$, for various types of varieties X:

$$\text{curves of genus } g: \qquad \begin{array}{cc} g & 1 \\ 1 & g \end{array} \tag{4}$$

$$\mathbb{P}^n: \qquad \begin{array}{cccc} 0 & 0 & \cdots & 1 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 1 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \end{array} \tag{5}$$

$$\text{abelian surfaces}: \qquad \begin{array}{ccc} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{array} \tag{6}$$

$$\text{K3 surfaces}: \qquad \begin{array}{ccc} 1 & 0 & 1 \\ 0 & 20 & 0 \\ 1 & 0 & 1 \end{array} \tag{7}$$

$$\mathbb{A}^n: \qquad \begin{array}{cccc} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \iddots & \vdots \\ 0 & 0 & \cdots & 0 \\ \infty & 0 & \cdots & 0 \end{array} \tag{8}$$

Note that $\mathbb{A}^n$ does not satisfy finiteness or Serre or Poincaré duality, as it isn't proper. In fact the Hodge diamond "knows" that $\mathbb{A}^n$ is "a contractible space with too many functions". Also

[3] In fact, there is a post somewhere on StackExchange or MathOverflow, which I can't find at the moment, showing that this symmetry propagates to the ith page of the spectral sequence for all $i \geq 1$.

note that all of these Hodge diamonds are symmetric across their diagonals, in addition to being rotationally symmetric. This is known as Hodge symmetry, and it is true over an arbitrary field of characteristic 0. It is not true in characteristic p, as we will see.

So algebraic de Rham cohomology can be computed (in some sense) by a process starting with the sheaf cohomology of each $\Omega^i_{X/k}$, and these can be calculated directly for most varieties we're interested in. In fact the situation is even better: if $\operatorname{char} k = 0$, the spectral sequence degenerates at E_1; that is, every map on and after the E_1 page is zero. In this case, $E_\infty = E_1$, and $h^i_{dR}(X/k)$ is just $\sum_{p+q=i} h^{p,q}(X/k)$. This can be proved in two ways. The first is to reduce to the case of $\mathbb{C}$ and appeal to classical (analytic) Hodge theory. The second is an algebraic proof of Deligne and Illusie, which remarkably proves a statement about characteristic zero using almost exclusively characteristic-p methods.

2.3 Miracles

There are a few things here that I view as miracles. The first is that all of these cohomology theories actually have the formal properties we expect of them. The second is that in spite of their vastly different origins, they behave quite similarly.

In fact, there are various theorems in p-adic Hodge theory that compare one cohomology theory to another after base change to a sufficiently large "period ring". There is also a web of conjectures, religious dogma, and some actual theorems (going by the name of "motives") predicting that all sufficiently well-behaved cohomology theories factor through some universal cohomology theory. I don't know anything about motives, but for the three cohomology theories we've discussed, there is now a known cohomology theory strong enough to recover all three. This is the so-called prismatic cohomology (formerly A_{inf}-cohomology) of Bhatt-Morrow-Scholze and Bhatt-Scholze, and it applies to the situation mentioned earlier where we have a smooth variety over a p-adic integer ring.

3 Serre's example over $\mathbb{F}_p$

We stated earlier that the Hodge-de Rham spectral sequence always degenerates on E_1 over a field of characteristic 0. In 1958, Serre gave an example showing that this does not hold in characteristic p. We'll also see that Serre's example fails Hodge symmetry, and the idea behind it will lead to even stranger behavior over p-adic integer rings.

Some vague motivation for Serre's construction: Imagine a p-fold covering of connected manifolds $Y \to X$, with Y simply connected. This gives $\pi_1(X) \cong \mathbb{Z}/p\mathbb{Z}$ and thus $H_1^{\text{sing}}(X;\mathbb{Z}) \cong (\mathbb{Z}/p\mathbb{Z})^{\text{ab}} = \mathbb{Z}/p\mathbb{Z}$. For any ring R, the universal coefficient theorem turns this into a copy of R/pR in $H^2_{\text{sing}}(X;R)$.

A cohomology theory valued in characteristic p (e.g. singular with $\mathbb{F}_p$ coefficients) will see this p-fold cover, but a cohomology theory valued in pure characteristic 0 (e.g. analytic de Rham or singular with $\mathbb{Q}$ coefficients) won't.

If we repeat the story for X, Y varieties over $\mathbb{F}_p$, then the Hodge cohomology $H^q(X, \Omega_X^p)$ will unsurprisingly behave like mod-p singular cohomology; i.e. we will have $h^{0,1} + h^{1,0} = 1$. But $H^i_{\mathrm{dR}}(X/\mathbb{F}_p)$ will behave like a Weil cohomology theory, like ℓ-adic étale cohomology, $\ell \neq p$. In particular, it will *not* see the p-fold cover! This forces the Hodge-de Rham spectral sequence to not converge on E^1.

Let's write down the actual construction. Let $p > 3$ be a prime, $k = \mathbb{F}_p$, and $G = \mathbb{Z}/p\mathbb{Z}$. This acts on $\mathbb{P}^3_k$ by the matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

(This is the only place where we use the condition $p > 3$: if $p \leq 3$, then $A^p \neq 1$.) Then consider the quotient $\mathbb{P}^3/G = \operatorname{Proj} k[w, x, y, z]^G$. This is a projective 3-fold, which is smooth away from the fixed locus of G on $\mathbb{P}^3$. But you can check that the fixed locus consists of a single point with projective coordinates $[1 : 0 : 0 : 0]$, so $\mathbb{P}^3/G$ has only one singularity, albeit a pretty nasty one.

Now we can build our varieties X and Y. Let X be a smooth surface in $\mathbb{P}^3/G$ that misses the singular point, and let Y be its preimage in $\mathbb{P}^3$. These are both smooth (connected) surfaces, and $Y \to X$ is a p-fold cover. Better yet, Y is cut out by a single homogeneous equation of degree d in $\mathbb{P}^3$, so we can calculate the sheaf cohomology of $\mathcal{O}_Y$ using the closed subscheme exact sequence. We can also get the cohomology of $\Omega^2_Y = \omega_Y$ by Poincaré duality, and that of Ω^1_Y by combining the closed subscheme, Euler, and conormal exact sequences. The upshot is that the Hodge diamond of Y looks like:

$$\begin{matrix} a & 0 & 1 \\ 0 & b & 0, \\ 1 & 0 & a \end{matrix} \tag{10}$$

where a and b are some explicit cubic polynomials in d.[4] One can then run a spectral sequence involving the group cohomology of G acting on the Hodge cohomology of Y to compute the Hodge diamond of X:

$$\begin{matrix} a/p + 1 & 0 & 1 \\ 1 & b/p & 1 \\ 1 & 0 & a/p + 1 \end{matrix}. \tag{11}$$

(Disclaimer: I worked this out almost completely a few years ago, but there may be some slight errors.) Here we can see that Hodge symmetry indeed fails. When one runs the Hodge-de Rham spectral sequence, two pairs of 1's cancel along knight's-move maps.

4 BMS's example over $\mathcal{O}_C$

[4]Namely, $a = \binom{d-1}{3}$ and $b = \frac{2d^3 - 6d^2 + 7d}{3}$.